Lecture 5. From large to small Reynolds numbers

Last lecture we saw that we can write N-S equations in terms of the dimensionless variables:

$$t' = \frac{tU}{L} \quad , \quad \underline{r}' = \frac{\underline{r}}{L} \quad ; \quad \underline{v}' = \frac{\underline{v}}{U}$$

where t, L, U are the characteristic time, length and velocity scales of the problem

defining the Reynold's nuber as $Re \equiv \frac{UL}{\nu}$, as the ratio of the $\frac{\text{inertial F's}}{\text{viscous F's}} = \frac{\frac{U^2\rho}{L}}{\frac{\eta U}{L^2}} = \frac{UL}{\nu}$ with $\nu = \frac{\eta}{\rho}$

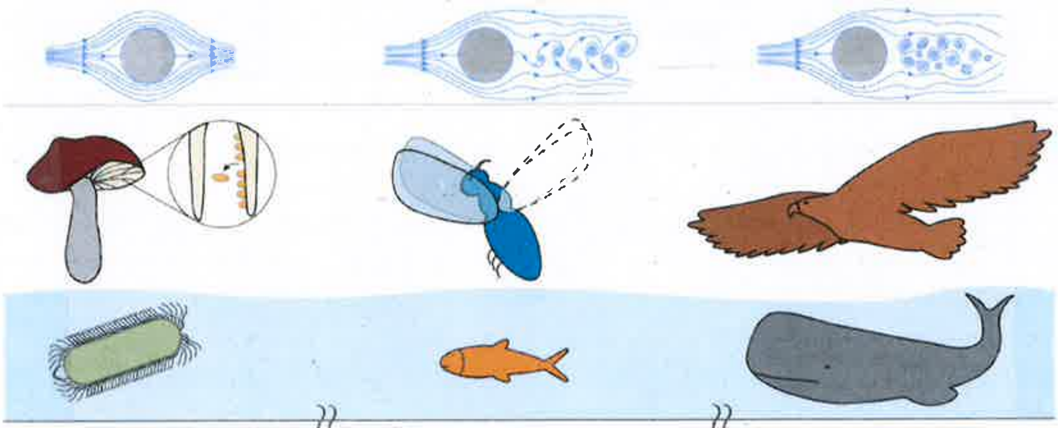
We have that N-S equation becomes:

$$Re\left[\frac{\partial \underline{v}'}{\partial t'} + \underline{v}' \cdot \nabla' \underline{v}'\right] = -\nabla' p' + \nabla'^2 \underline{v}' \quad \text{where } p' = \frac{pL}{\rho U \nu}$$

Or alternatively:

$$\frac{\partial \underline{v}'}{\partial t'} + \underline{v}' \cdot \nabla' \underline{v}' = -\nabla' p' + \frac{1}{Re} \nabla'^2 \underline{v}' \quad \text{where } p' = \frac{p}{\rho U^2} \quad (*)$$

The Reynold's number is the parameter dominating the dynamics of viscous flows.



Video #7. 4:48 - 5:10 $\quad Re = 0.05, \ 10, \ 200, \ 3000$

Very large Reynold's numbers.

Looking at our equation of motion (*) we can see that if $Re \gg 1$, it reduces to

$$\frac{\partial \underline{v}'}{\partial t'} + \underline{v}' \cdot \nabla' \underline{v}' = -\nabla' p' \quad \text{or considering a steady flow}$$

$$\underline{v}' \cdot \nabla' \underline{v} = -\nabla' p' \quad \text{or dimensionally} \quad \rho(\underline{v} \cdot \nabla)\underline{v} = -\nabla p \leftarrow$$

Recall this is Euler's equation for inviscid fluids.

This equation can be solved if we consider the fluid has no vorticity (swirling motion). In this case, if we assume no vorticity:

$$\nabla \times \underline{v} = 0 \ ; \ \text{we can propose a solution } \underline{v} = \nabla \varphi.$$

With this the incompressibility condition $\nabla \cdot \underline{v} = 0 \Rightarrow \nabla^2 \varphi = 0$ Laplace's equation

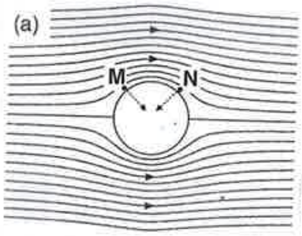
To solve Euler's equation we use the identity: $(\underline{v}\cdot\nabla)\underline{v} = \nabla\left(\frac{1}{2}|\underline{v}|^2\right) - \underline{v}\times\left(\cancel{\nabla\times\underline{v}}^{\,0}\right)$
And considering $\underline{v} = \nabla\varphi$

$$\rho\left[\nabla\left(\frac{1}{2}\right)|\nabla\varphi|^2\right] = -\nabla p \quad \Rightarrow \nabla\left(\frac{1}{2}|\nabla\varphi|^2 + \frac{p}{\rho}\right) = 0$$

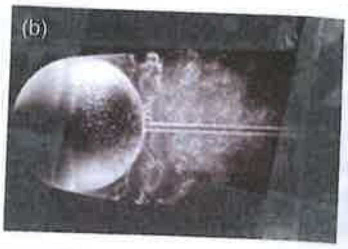
$$\Rightarrow \frac{1}{2}|\nabla\varphi|^2 + \frac{p}{\rho} = \text{constant}$$ ← Bernoulli's equation (which we had previously derived using energy conservation arguments)

If we know what the constant is anywhere in the fluid, then ~~we tell~~ wherever the speed of the fluid is known, so is the pressure, and therefore the drag on a rigid body can be evaluated. The drag on a rigid body in a perfect fluid ~~by~~ was computed by D'Alembert.

D'Alembert's paradox: The drag on a rigid body moving at constant speed in a perfect fluid is always zero. ! Goes against everything we know in real life.



We can get some intuition on D'Alembert's paradox by considering a sphere moving at steady speed through a perfect fluid.
The sphere experiences an incoming flow shown by the streamlines in the figure on the left. To calculate the drag we need to know the traction on the surface of the sphere. (F/A). In this case there is no viscosity so the only traction is ~~due~~ to the normal stress due to pressure.
There is a perfect front-back symmetry in the fluid flow "before" and "after" the sphere. Specifically the magnitude of the flow speed at e.g. point M, will be the same as at N. Since the speeds are the same, Bernoulli implies the pressures at the same. At each point the pressure exerts a stress in a direction ⊥ to the sphere (shown by the arrows). Around the sphere the tractions will cancel out (for every traction there is an equal + opposite on the other side).
$\Rightarrow$ sum of all tractions = total drag = 0.



This is an actual picture of the flow around a rigid sphere at $Re = 15{,}000$, which is clearly different from the flow obtained by neglecting viscosity.

We also know that in real life we do experience drag e.g. when swimming, riding a bike, etc.

What went wrong?
No matter how large Re is, the effects of viscosity somehow still matter.

Mathematically what went wrong:

$$\frac{\partial \underline{v}'}{\partial t'} + \underline{v}'\cdot\nabla'\underline{v}' = -\nabla' p' + \cancel{\frac{1}{Re}\nabla'^2\underline{v}'}^{\,0}$$ we got rid of the second derivative

$\Rightarrow$ Now we consider this equation using only one B-C. We can solve the Euler eq. of motion by using only the impermeability condition $(\underline{v}\cdot\hat{n} = \underline{V}\cdot\hat{n})$
↑ vel fluid at wall ↑ vel of wall

We have discarded the no-slip B-C by neglecting viscosity [we have discussed the no-slip B-C is a consequence of the action of viscosity].

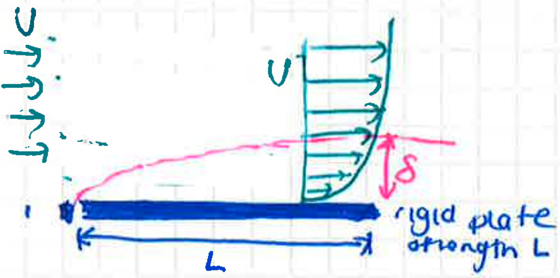
Boundary layers

~~As we~~ Removing viscosity implies removing the no-slip B-C, so the fluid is allowed to slip on the surface of the body.

From a physical standpoint assuming vanishing viscosity means removing the source of friction in the fluid. However, that microscopic friction is responsible for the velocity being zero on a solid boundary. It might be true that viscosity is not very important in most of the fluid but it matters a lot near the surface of the body (boundary).

Prandtl showed that the effect of viscosity is confined to thin boundary layers near rigid surfaces. Outside of these regions the flow behaves like a perfect fluid. Inside the boundary layer things are very different and the flow velocity has to be brought down to zero quickly on the surface to satisfy the no-slip condition.

The size of a boundary layer



Consider a thin plate in a flow of speed U
let L be the length of the plate
w the width of the plate
We want to know δ, which is the size of the boundary layer.

We'll estimate the boundary layer size by applying Newton's second law and balancing the rate at which momentum is lost by the viscous force. (Viscous stresses all occur at the boundary layer)

The momentum in the boundary layer is $\sim MU$ where M = mass of fluid in the boundary layer.

In terms of density $M \sim \rho V$ where V is the volume of fluid of the boundary layer $\Rightarrow M \sim \rho L w \delta$

$\therefore$ momentum of boundary layer $\sim \rho L w \delta U$

Changes in the boundary layer (driven by the flow of speed U) along the plate occur in the flow timescale

$$T \sim \frac{L}{U}$$

Now, so the rate at which momentum changes in the boundary layer is:

$$\frac{dp}{dt} \sim \frac{\rho L w \delta U}{T} = \rho \delta w U^2 \quad (*)$$

This rate of change in momentum should be equal to the viscous forces. Inside the boundary layer the viscous shear stress τ, i.e. the tangential component of the viscous stress, caused by a velocity gradient parallel to the surface is:

$$\tau = \eta \frac{dv}{dz} \Rightarrow \tau \sim \eta \frac{U}{\delta}$$

Since the viscous shear stress is a force / unit area, the total viscous force exerted by the plate on the fluid has magnitude:

$$F \sim \tau L w = \frac{\eta L w U}{\delta} \quad (**)$$

Now we are ready to apply Newton's second law to balance the rate at which momentum is lost by the total viscous force (setting $(*) = (**)$):

$$\rho \delta w U^2 \sim \frac{\eta L w U}{\delta} \Rightarrow \frac{\delta^2}{L^2} \sim \frac{\eta}{\rho U L} \Rightarrow \boxed{\delta \sim \frac{L}{\sqrt{Re}}}$$ size of the boundary layer

This inverse square root scaling shows that for $Re \gg 1$ the boundary layer is much thinner than the size of the plate, $\delta \ll L$. The fluid speeds near the plate undergo changes over very small distances, which leads to considerable shear stresses and drag.

Flow separation and wake

We have not discussed what is the flow pressure in the boundary layer. The pressure in the boundary layer is the same as the pessure outside it. In the case we discussed of the plate, flow has a constant speed and pressure so there is no pressure gradient in the boundary layer.

However in flow around more complex ~~surfaces~~ shapes ~~(e.g.)~~ the fluid dynamics inside the boundary layer ~~become~~ change because of the viscous stresses + the pressures inherited from the outside flow.

Changes in pessure along the body can be, → favorable: On their own they would induce fluid motion in the same direction as the actual flow

~~unfavorable~~:
adverse: when they "push" in the opposite direction of the flow.

If the pressure gradient is adverse ~~for~~ for a long time it will slow down the flow in the boundary layer ~~&~~ until it stops causing its detachment!! from the body! This phenomenon is called flow separation. If this occurs a wake appears associated with a steep increase in drag on the body.

Low Reynolds number flows

When $Re \ll 1$ viscous forces dominate over inertia so much that we can neglect the role of inertia. Dropping the inertial term in N-S equations we are left with:

$$\boxed{\begin{aligned} \nabla p &= \nu \nabla^2 \underline{v} \\ \nabla \cdot \underline{v} &= 0 \end{aligned}}$$

"Stokes equations" for incompressible Newtonian fluid

-It is possible to show rigorously that solutions to the Stokes equations with prescribed velocity boundary conditions are unique. As a consequence if a flow solution for a particular set up is obtained using any mathematical method or guess work, then it is the unique solution to the problem.

-The velocity field is divergence free

$$\nabla \cdot (\nabla p) = \nabla \cdot (\nu \nabla^2 \underline{v}) = \nabla^2 (\nu \cancelto{0}{\nabla \cdot \underline{v}})$$

$$\Rightarrow \nabla^2 p = 0$$ Pressure satisfies Laplace's equation → harmonic function.

As a result computing the laplacian of stokes equation gives:

$$\nabla^4 \underline{v} = \nabla^2 \nabla^2 \underline{v} = 0$$ ← velocity field is biharmonic

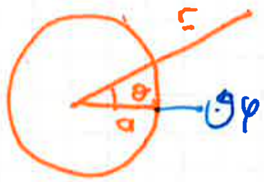
Taking the curl of the stokes equation yields:

$$\cancelto{0}{\nabla \times (\nabla p)} = \nabla \times (\nu \nabla^2 \underline{v}) = \nu \nabla^2 (\cancelto{\underline{w}}{\nabla \times \underline{v}})$$

$$\Rightarrow \nabla^2 \underline{w} = 0$$ → the vorticity of a Stokes flow is also harmonic.

Stokes flow past a sphere

One of the most famous solutions to stokes's equation is flow past a sphere. It leads to the relationship between the velocity and the drag used to measure the viscosity of fluids, by observing the flow rate of a sphere through them.



Consider a sphere of radius a at rest.

$\underline{v} = \underline{v}(v_r, v_\theta)$, due to symmetry $v_\varphi = 0$

The boundary conditions are:

$v_r = v_\theta = 0$ at $r = a$

$v_r \to v_0 \cos\theta$, $v_\theta \to -v_0 \sin\theta$ as $r \to \infty$

where v_0 is the free stream velocity

The equations of motion in spherical coordinates become:

$$\frac{\partial v_r}{\partial r} + \frac{2v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta \cot\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} = 0 \quad \leftarrow \text{continuity}$$

$$0 = -\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} - \frac{2v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \varphi^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}\right]$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \varphi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}\right]$$

$$0 = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi} + \nu\left[\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\varphi}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \varphi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \varphi}\right]$$

Considering $\underline{v} = \underline{v}(v_r, v_\theta)$, same for p and $\frac{\partial}{\partial\varphi} = 0$ we have:

$$\frac{\partial v_r}{\partial r} + \frac{2v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta \cot\theta}{r} = 0 \quad \text{continuity}$$

$$0 = -\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} - \frac{2v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta\cot\theta}{r^2}\right]$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

The solutions to these equations are:

$$v_r = v_0\cos\theta\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right]$$

$$v_\theta = -v_0\sin\theta\left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right]$$

$$p - p_0 = -\frac{3}{2}\frac{\nu v_0 a}{r^2}\cos\theta$$

The force/unit area in the flow direction acting on a point on the surface of the sphere is the sum of the appropriate components of the viscous & pressure forces:

$$\sigma = -\nu \left(\frac{\partial v_\theta}{\partial r}\right|_{r=a} \sin\theta - (p-p_0)\Big|_{r=a} \cos\theta = \frac{3\nu v_0}{2a}$$

subs the solution v_θ

Since this is independent of θ, the total force on the sphere is just $\sigma \cdot A_{sphere}$

$$\rightarrow \boxed{F_D = 4\pi a^2 \sigma = 6\pi \eta a v_0 \quad \text{Stokes drag on a sphere}}$$

Stokeslet and the Oseen tensor

We can generalize Stokes equations to include an external force:

$$-\nabla p + \nu \nabla^2 \underline{v} + \underline{f}_{ext} = 0$$

$$\nabla \cdot \underline{v} = 0$$

These equations can be solved for any constant force $\underline{f}_{ext}$ in the case that the fluid fills all ~~taken~~ of space and both the fluid flow and the pressure are zero at infinity

Because we have a system of linear equations, if we know the solution of a point force at any point in space, we can find the solution for any combination of point forces by simply adding them, and for any continuous force distribution by integrating over point forces.

From the theory of differential equations, the function that gives the solution in terms of such a point force is known as the Green's function, for the case of Stokes' equations its know as a Stokeslet, we can take the origin to coincide to the point where our force is and write:

$$\underline{f}_{ext}(\underline{r}) = \underline{F}\, \delta(\underline{r})$$

here $\underline{F}$ is an ordinary force, i.e. not a force/unit volume since $[\delta(\underline{r})] = 1/\text{Volume}$. The fluid flow field and the pressure resulting from a point force are given by:

$$\underline{v}(\underline{r}) = \frac{1}{8\pi\nu r}\left(\underline{F} + \frac{\underline{F}\cdot\underline{r}}{r^2}\,\underline{r}\right)$$

$$p(\underline{r}) = \frac{\underline{F}\cdot\underline{r}}{4\pi r^3}$$

We can write the equation in component form separating out the magnitude of the force:

$$v_i = \frac{1}{8\pi\nu r}\left(\delta_{ij} + \frac{r_i r_j}{r^2}\right) F_j = J_{ij}(\underline{r})\, F_j$$

where J_{ij} is the Oseen tensor

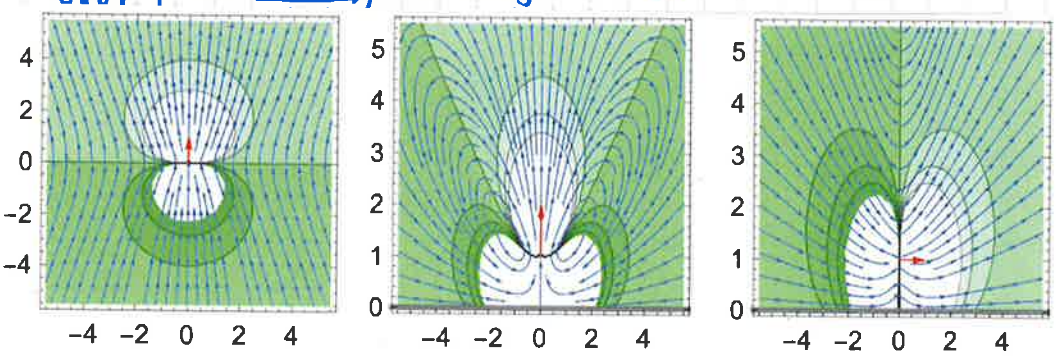


Figure 3.9: Flow due to a Stokeslet (point force). In all three figures, the red arrow indicates the point force, the blue arrows the resulting flow field, and the green contours the lines of constant pressure. (a) Free space. (b) Stokeslet above a wall, pointing away from the wall. (c) Stokeslet above a wall, pointing along the wall.

As explained before, to find the flow field for ~~of~~ multiple forces we add individual forces. ~~For Laccas~~ To find the pressure & velocity of a continuous force distribution $\underline{f}(\underline{r})$ we integrate:

$$\underline{v}(\underline{r}) = \int \underline{\underline{I}}(\underline{r}-\underline{r}')\cdot \underline{f}(\underline{r}')\,d\underline{r}'$$

$$p(\underline{r}) = \int \frac{\underline{f}(\underline{r}')\cdot(\underline{r}-\underline{r}')}{4\pi|\underline{r}-\underline{r}'|^3}\,d\underline{r}'$$

Swimming at Low Reynolds number & the scallop theorem

To get insight into some key properties of Low Re flows we'll consider the dimensionless stokes equations with scaling $p' = \frac{pL}{\mu V}$ in the absence of external forces.

$$-\nabla' p' + \nabla'^2 \underline{v}' = 0$$

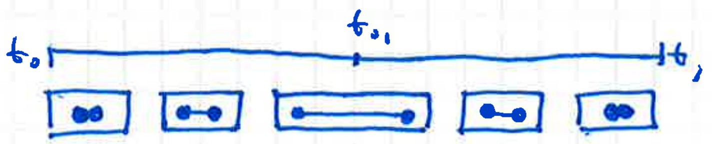
* At every point in the fluid viscous stresses balance pressure ⇒ Net force = 0
* There is no explicit parameter (time or vel scale), so changing the rate at which deformation occurs does not change the flow pattern (i.e changing V only scales the solution)
  ⇒ Rate independence: Flow pattern only depends on geometry
* If $\underline{v}$ and p are solutions to the equation, when I change $\underline{v}$ to $-\underline{v}$ the p will change to $-p$ (only reverses sign)
  ⇒ Kinematic reversibility: Kinematic pattern is identical in forward and backward motion → time reversibility

Video #7 13:16 – 14:50

The scallop theorem: "If an animal tries to swim by reciprocal motion in Low Re, it won't go anywhere"

A scallop at low Re can't swim because it only has one hinge, if you only have one degree of freedom you are bound to make a reciprocal motion.

Consider a swimmer undergoing a periodic shape deformation on the time interval $[t_0, t_1]$.



Periodic sequence of representative shapes in the time interval $[t_0, t_1]$

We can split the interval $[t_0, t_1]$ into:

$$[t_0, t_{01}] \cup [t_{01}, t_1]$$

All the shapes between $[t_0, t_{01}]$ are mapped into $[t_{01}, t_1]$ as shown in the schematic above. We will now compute the total displacement traveled by the swimmer $\Delta_{t_0\to t_1}$.

By construction we have:

$$\Delta_{t_0\to t_1} = \Delta_{t_0\to t_{01}} + \Delta_{t_{01}\to t_1} \quad (*)$$

Because of kinematic reversibility we have that: $\Delta_{t_{01}\to t_1} = -\Delta_{t_1\to t_{01}}$ (**)

From the rate independence property: $\Delta_{t_1\to t_{01}} = \Delta_{t_0\to t_{01}}$ since they go through the same sequence of shapes.

So subs this in (*): $\Delta_{t_0\to t_1} = \Delta_{t_1\to t_{01}} + \underbrace{\Delta_{t_{01}\to t_1}}_{\text{Due to (**)}} = \Delta_{t_1\to t_{01}} - \Delta_{t_1\to t_{01}} = 0$

A reciprocal swimmer is a non-swimmer.